Section 3.3, begin 3.4

Note Title 2/10/2008

We left off having shown by the SQUEEZE THEOREM that

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

meant $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

And it really is: eg

$$\frac{\sin(0.1)}{0.1} \doteq 0.99833$$

$$\frac{\sin(0.000734)}{0.000734} \doteq 0.99999991020733$$

$$\frac{\sin(-0.0000000082468246)}{-0.0000000082468246}$$

$$\doteq 1 - 1.133 \cdot 10^{-15}$$

really quite accurate.

This was HALF the story:

$$\lim_{\Delta x \to 0} \frac{\sin(a+\Delta x) - \sin(a)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(a)\cos\Delta x + \sin(\Delta x)\cos a - \sin(a)}{\Delta x}$$

$$= \sin(a) \lim_{\Delta x \to 0} \frac{\cos\Delta x - 1}{\Delta x}$$

$$+ \cos(a) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

and we just showed that second limit to be 1. What about the first?

$$\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos^2(\Delta x) - 1}{\Delta x\,(\cos\Delta x + 1)}$$

trick!

$$= \lim_{\Delta x \to 0} \frac{-\sin^2 \Delta x}{\Delta x (\cos \Delta x + 1)}$$

$$= \lim_{\Delta x \to 0} \left(\frac{+\sin \Delta x}{\Delta x} \right) \cdot \frac{(-\sin \Delta x)}{(\cos \Delta x + 1)}$$

and both

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$$

and

$$\lim_{\Delta x \to 0} \frac{-\sin \Delta x}{\cos \Delta x + 1} = \frac{0}{\cos(0) + 1} = \frac{0}{2}$$

$$= 0$$

exist, so the limit of the product is the product of the limits:

$$\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = 0.$$

$$\therefore \lim_{\Delta x \to 0} \frac{\sin(a+\Delta x) - \sin(a)}{\Delta x}$$

$$= \sin(a) \cdot 0 + \cos(a) \cdot 1$$

$$= \cos(a).$$

$$\boxed{\therefore \text{ if } y = \sin(x) \text{ then } \frac{dy}{dx} = \cos(x).}$$

What is $\frac{d}{dx}\cos(x)$?

Same again:

$$\lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x)\cos\Delta x - \sin(x)\sin\Delta x - \cos(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \cos(x) \frac{(\cos \Delta x - 1)}{\Delta x} - \sin(x) \frac{\sin \Delta x}{\Delta x}$$

$$= \cos(x) \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} - \sin(x) \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}$$

$$= \cos(x) \cdot 0 - \sin(x) \cdot 1$$

$$= -\sin(x).$$

$$\boxed{\therefore \frac{d}{dx} \cos(x) = -\sin(x).}$$

Use Quotient Rule to find

$$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x}$$

$$= \frac{\left[\frac{d}{dx} \sin(x)\right] \cos x - \sin x \frac{d}{dx} \cos x}{\cos^2 x}$$

$$= \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x .$$

Exercise: derive all the rules in the box on p. 193.

CHAIN RULE : Section 3.4.

We will use Caratheodory's method, as a complement to the treatment in the book.

We are going to find

$$(F \circ G)'(x) = \frac{d}{dx}\left(F(G(x))\right) .$$

Carathéodory's forms:

$$G(x) = G(a) + g_a(x)(x-a)$$

$$F(x) = F(G(a)) + f_G(x)(x - G(a))$$

trick!

where $g_a(x)$ is cts at $x=a$ and

$$g_a(a) = G'(a)$$

and where $f_G(x)$ is cts at

$x = G(a)$ and

$$f_G(G(a)) = F'(G(a)).$$

Then

$$F(G(x)) = F(G(a)) + f_G(G(x))(G(x) - G(a))$$

$f_G(G(x))$ is a composition of cts functions & thus cts.

we have also

$$G(x) - G(a) = g_a(x)(x-a).$$

Therefore

$$F(G(x)) = F(G(a)) + \underbrace{f_G(G(x))\, g_a(x)}\,(x-a)$$

and this function is cts and of the required form.

its value at $x=a$ is

$$f_G(G(a))\, g_a(a)$$

$$= F'(G(a))\, G'(a).$$

This is the chain rule.

To compare this with the proof on p 203:

the book uses the letter

b to mean $G(a)$

$g(x)$ where I use $G(x)$

$f(x)$ where I use $F(x)$

and hides some information under "ε_1" and "ε_2"

But fundamentally both proofs say the same thing and use the same ideas.

(Carathéodory's is just cleaner)

CHAIN RULE

$$(F \circ G)'(x)$$

$$= (F' \circ G)(x) \cdot G'(x) .$$

OR

$$\frac{d}{dx}\left(F(G(x))\right) = F'(G(x))\, G'(x)$$

"differentiate outside fn
& multiply by derivative
of inside fn"

<u>Example</u>

$$F(x) = e^x$$

$$G(x) = \ln(x)$$

$$(F \circ G)(x) = e^{\ln(x)}$$

$$(F \circ G)'(x) = F'(G(x))\, G'(x)$$

$$F'(x) = e^x$$

$$G'(x) = \frac{1}{x}$$

$$\therefore F'(G(x)) \cdot G'(x)$$

$$= e^{\ln x} \cdot \frac{1}{x}.$$

We may simplify:

$$(F \circ G)(x) = e^{\ln x}$$

$$= x$$

by definition.

$$\boxed{\begin{aligned} &\lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\ln\left(\frac{x+\Delta x}{x}\right)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x} \\ &= \frac{1}{x} \lim_{\Delta x \to 0} \frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\Delta x / x} \\ &= \frac{1}{x} \cdot 1 \end{aligned}}$$

Our answer above becomes

$$e^{\ln x} \cdot \frac{1}{x} = x \cdot \frac{1}{x} = 1$$ which is also right.